工程數學 Final Exam Dec 17, 2024

ID:________________ Name:________________

滿分 30 分

(1) Solve this ODE by Cauchy-Euler Equation: (5points)

$$4x^2y'' + 8xy' + y = 0$$

(2) Solve this ODE by Cauchy-Euler Equation: (5points)

$$x^2y'' - 2y = 6x$$

(3) Inverse Laplace transform: (3 points)

$$\mathcal{L}^{-1}\left\{\frac{3s-1}{s^2-2s+5}\right\} =$$

(4) Solve this ODE by Laplace transform: (5points)

$$y' + 4y = e^{-2t}\ , \qquad y(0) = 1$$

(5) Laplace transform of derivative, solve the ODE: (5points)

$$y' - 6y = 0 \qquad y(-1) = 4$$

(6) Inverse Laplace transform: (2points)

$$\mathcal{L}^{-1}\left\{\frac{-5e^{-5s}}{(s+2)^2} + \frac{1}{s}\right\} =$$

(7) find the Fourier series of the function f: (5 points)

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 \le x < \pi \end{cases}\ , \quad f(x+2\pi) = f(x)$$

公式表:

Auxiliary equation: $m^2 + am + b = 0$　根為: $\frac{-a\pm\sqrt{a^2-4b}}{2}$

$\sqrt{a^2-4b}$ 解有三種可能:

<1>相異實根: m_1, m_2

ODE 通解為: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

<2>實數重根: m

ODE 通解為: $y = (c_1 + c_2 x)e^{mx}$

<3>複數根 (共軛虛根): $\alpha \pm \omega i$

ODE 通解為: $y = e^{\alpha x}(A\cos\omega x + B\sin\omega x)$

求 y_p:

Undetermined coefficients method (未定係數法):

R(x)	y_p 假設型
k	A
e^{ax}	Ae^{ax}
cos bx 或 sin bx	A cos bx + B sin bx
x^n	$A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x^1 + A_0$
cx^n	$A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x^1 + A_0$
$x^n e^{nx}$	$e^{nx}(A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x^1 + A_0)$
$cx^n e^{nx}$	$e^{nx}(A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x^1 + A_0)$
$x^n\cos bx$ 或 $x^n\sin bx$	$(A_n x^n + A_{n-1}x^{n-1} + \dots + A_1 x^1 + A_0)\cos bx$ + $(B_n x^n + B_{n-1}x^{n-1} + \dots + B_1 x^1 + B_0)\sin bx$

註: a, b, c, k, n, A, B 為常數

- Cauchy-Euler equation:
- 令 $t = \ln x$ (以 t 取代 x,　$x = e^t$)

$xy' = D_t y$

$x^2 y'' = D_t(D_t - 1)y$

$x^3 y''' = D_t(D_t - 1)(D_t - 2)y$

- Laplace Transform

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2-a^2}$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2-a^2}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\frac{1}{s-a}$	12	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

First Shifting Theorem, *s*-Shifting

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$$

Laplace Transform of Derivatives

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Laplace Transform of Integral 積分轉換

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

Second Shifting Theorem; Time Shifting	**Examples for *t*-shifting by 1**
$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$ $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $f(t-a)u(t-a) = \mathcal{L}^{-1}\{e^{-as}F(s)\}$	$1 \mapsto \frac{1}{s}$ $u(t-1) \mapsto e^{-s}\frac{1}{s}$ $t \mapsto \frac{1}{s^2}$ $(t-1)u(t-1) \mapsto e^{-s}\frac{1}{s^2}$ $t^2 \mapsto \frac{2}{s^3}$ $(t-1)^2u(t-1) \mapsto e^{-s}\frac{2}{s^3}$ $\sin t \mapsto \frac{1}{s^2+1}$ $\sin(t-1)u(t-1) \mapsto e^{-s}\frac{1}{s^2+1}$

傅立葉級數 ***(週期 2L)***	**Fourier Cosine Series** (*even function, 偶函數*) ***(週期 2L)***
$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$ **(a)** $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ **(b)** $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$ $n = 1, 2, \cdots$ **(c)** $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$ $n = 1, 2, \cdots$	$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$ $(f \text{ even})$ $a_0 = \frac{1}{L}\int_0^L f(x)\,dx,$ $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,$ $n = 1, 2, \cdots$ **Fourier Sine Series** (***odd function, 奇函數***) ***(週期 2L)*** $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$ $(f \text{ odd})$ $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$\sin\pi = 0, \quad \sin 2\pi = 0, \quad \sin 3\pi = 0, \ \ldots$
$\cos\pi = -1, \quad \cos 2\pi = 1, \quad \cos 3\pi = -1, \ \ldots$