工程數學 Final Exam Dec 19, 2023

ID:________________ Name:________________

滿分 30 分

(1) Using Cauchy-Euler Equation to solve the ODE. (4 Points)

$x^2y'' + 2xy' - 2y = 0$

(2) Using Laplace transforms to solve the ODE. (4 Points)

$y' - 6y = 0 \quad y(-1) = 4$

(3) Find $\mathcal{L}^{-1}\{F\}$. (5 Points)

$$F(s) = \frac{5s - 95}{s^2 + 2s + 101}$$

(4) Find $\mathcal{L}^{-1}\{F\}$. (2 Points)

$$F(s) = \frac{1}{s^2} - \frac{48}{s^5}$$

(5) Find $\mathcal{L}^{-1}\{F\}$. (2 Points)

$$F(s) = \frac{4s}{4s^2 + 1}$$

(6) Using Laplace transforms to solve the ODE. (3 Points)

$y' - y = 1$, $y(0) = 0$

(7) Find $\mathcal{L}^{-1}\{F\}$. (3 Points)

$$F(s) = \frac{e^{-3s}}{(s+2)^2}$$

(8) Find $\mathcal{L}^{-1}\{F\}$. (2 Points)

$$F(s) = \frac{e^{-s}}{s^2+\pi^2}$$

(9) Find the Fourier series of the function $f(x)$. (5 Points)

$f(x) = x + \pi, \quad -\pi < x < \pi, \; f(x + 2\pi) = f(x)$

公式表:

Auxiliary equation: $m^2 + am + b = 0$ 根為: $\frac{-a\pm\sqrt{a^2-4b}}{2}$

$\sqrt{a^2-4b}$解有三種可能:

<1>相異實根: m_1, m_2

ODE 通解為: $y = c_1e^{m_1x} + c_2e^{m_2x}$

<2>實數重根: m

ODE 通解為: $y = (c_1 + c_2x)e^{mx}$

<3>複數根 (共軛虛根): $\alpha \pm \omega i$

ODE 通解為: $y = e^{\alpha x}(A\cos\omega x + B\sin\omega x)$

求 y_p:

Undetermined coefficients method (未定係數法):

R(x)	y_p假設型
k	A
e^{ax}	Ae^{ax}
cos bx 或 sin bx	A cos bx + B sin bx
x^n	$A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x^1 + A_0$
cx^n	$A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x^1 + A_0$
x^ne^{nx}	$e^{nx}(A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x^1 + A_0)$
cx^ne^{nx}	$e^{nx}(A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x^1 + A_0)$
$x^n\cos bx$ 或 $x^n\sin bx$	$(A_nx^n + A_{n-1}x^{n-1} + \ldots + A_1x^1 + A_0)\cos bx$ + $(B_nx^n + B_{n-1}x^{n-1} + \ldots + B_1x^1 + B_0)\sin bx$

註: a, b, c, k, n, A, B 為常數

- Cauchy-Euler equation:

  令 t =ln x (以 t 取代 x, $x = e^t$)

  $xy' = D_ty$

  $x^2y'' = D_t(D_t-1)y$

  $x^3y''' = D_t(D_t-1)(D_t-2)y$

- Laplace Transform

	$f(t)$	$\mathscr{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$

	$f(t)$	$\mathscr{L}(f)$
7	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
8	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\frac{s}{s^2-a^2}$
10	$\sinh at$	$\frac{a}{s^2-a^2}$
11	$e^{at}\cos\omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at}\sin\omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

First Shifting Theorem, *s*-Shifting

$\mathscr{L}\{e^{at}f(t)\} = F(s-a)$

$e^{at}f(t) = \mathscr{L}^{-1}\{F(s-a)\}$

Laplace Transform of Derivatives

$$\mathscr{L}(f') = s\mathscr{L}(f) - f(0)$$

$$\mathscr{L}(f'') = s^2\mathscr{L}(f) - sf(0) - f'(0)$$

$$\mathscr{L}(f^{(n)}) = s^n\mathscr{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$$

Laplace Transform of Integral 積分轉換

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s), \qquad \text{thus} \qquad \int_0^t f(\tau)\,d\tau = \mathscr{L}^{-1}\left\{\frac{1}{s}F(s)\right\}$$

Second Shifting Theorem; Time Shifting	**Examples for *t*-shifting by 1**
$\tilde{f}(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{if } t < a \\ f(t-a) & \text{if } t > a \end{cases}$ $\mathscr{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$ $f(t-a)u(t-a) = \mathscr{L}^{-1}\{e^{-as}F(s)\}$	$1 \mapsto \frac{1}{s}$ $u(t-1) \mapsto e^{-s}\frac{1}{s}$ $t \mapsto \frac{1}{s^2}$ $(t-1)u(t-1) \mapsto e^{-s}\frac{1}{s^2}$ $t^2 \mapsto \frac{2}{s^3}$ $(t-1)^2u(t-1) \mapsto e^{-s}\frac{2}{s^3}$ $\sin t \mapsto \frac{1}{s^2+1}$ $\sin(t-1)u(t-1) \mapsto e^{-s}\frac{1}{s^2+1}$

傅立葉級數 ***(週期 2L)***	**Fourier Cosine Series** (***even function, 偶函數***) ***(週期 2L)***
$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$ **(a)** $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$ **(b)** $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots$ **(c)** $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad n = 1, 2, \cdots$	$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad (f \text{ even})$ $a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, \cdots$ **Fourier Sine Series** (***odd function, 奇函數***) ***(週期 2L)*** $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad (f \text{ odd})$ $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$\sin\pi = 0, \quad \sin 2\pi = 0, \quad \sin 3\pi = 0, \ \ldots$
$\cos\pi = -1, \quad \cos 2\pi = 1, \quad \cos 3\pi = -1, \ \ldots$